Revisiting the *1/L* Problem in Rheological Models for Time-Domain Seismic Wave Propagation

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SUMMARY

After the work of Liu, Anderson and Kanamori (Liu 76) on the general standard linear solid to give a realistic description of the attenuation of the Earth, later works (Carcione 07, Moczo 05, Cao 14) in literature have frequently mentioned that there is an error of missing *1/L* factor in equations in Liu *et. al.*'s paper. Here *L* is the number of linear standard solids used in general standard linear solid. We revisit this issue and point out that, this so-called missing *1/L* factor is originated from different definitions of the stress relaxation time.

INTRODUCTION

The anelastic losses need to be included in the seismic wave propagation simulation. Furthermore, observations have shown that, these attenuations can be described by a nearly frequency-independent *Q* factor over the seismic frequency range (McDonal 58, Liu 76, Dahlem 98). In time domain, the incorporation of attenuation is usually described by a rheological model that establishes a time convolutional relation between the stress and strain. Liu *et al.* (Liu 76) showed that a linear rheological model based on the general standard linear solid (GSLS) can give a realistic description of the observed frequency-independent *Q* factor. However, later publication (Carcione 07,Moczo 05,Cao 14) frequently mentions that there is an error of missing *1/L* factor in the equations Liu et al's paper, where *L* is the number of linear standard solids used in GSLS. For example, Moczo and Kristek mentioned in their paper (Moczo 05), "*Note that Liu et al. [1976], in generalizing the strain-stress relation for one ZB (equation 16 in their paper) to the relation for the GZB (equation 22 in their paper), introduced* ***an error****, which then has been repeated in the following papers dealing with the incorporation of the attenuation based on the GZB - even after Carcione [2001] published correct formulas for the relaxation function and modulus. In all papers we found, there is the same error – the missing factor 1/L in the viscoelastic modulus and relaxation function*". We revisit this so-called missing "*1/L*" problem in the rheological models used in time-domain seismic wave propagation, and we find that this *1/L* factor arises because of different definitions of relaxation times, and therefore, there is no such an error of missing *1/L* factor in Liu *et al*'s work.

THE MISSING 1/L FACTOR ISSUE

The time-dependent relation between stress σ and strain ε (the tensor property of the stress and strain is irrelevant here so we can treat all quantities as scalars) in a viscoelastic medium is given by the Boltzmann principle

$$\sigma(t) = \partial_t \psi(t) * \varepsilon(t) = M(t) * \varepsilon(t) \tag{1}$$

where $\psi(t)$ is the relaxation function and $M(t)=\partial_t\,\psi(t)$ the modulus function respectively, and * denotes time convolution. Given the viscoelastic modulus, the quality factor $Q(\omega)$ is

$$Q(\omega) = \frac{\operatorname{Re} M(\omega)}{\operatorname{Im} M(\omega)}, \tag{2}$$

where $M(\omega)$ is the Fourier transformation of $M(t)$ and ω is the angular frequency.

Liu *et al.* (Liu 76) constructed the modulus $M(\omega)$ from a linear rheological model composed of a number of *L* standard linear solids, which is called generalize Zener body, as shown in Fig. 1a. They first considered a linear rheological model with a single standard linear solid, and obtained the $M(\omega)$, $Q(\omega)$ and $\psi(t)$ as

$$M(\omega) = M_R \frac{1 + i\omega\tau_\varepsilon}{1 + i\omega\tau_\sigma}, \tag{3}$$

$$Q(\omega) = \frac{1 + \dfrac{\omega^2\tau_\sigma(\tau_\varepsilon - \tau_\sigma)}{1 + \omega^2\tau_\sigma^2}}{\dfrac{\omega\tau(\tau_\varepsilon - \tau_\sigma)}{1 + \omega^2\tau_\sigma^2}}, \tag{4}$$

$$\psi(t) = M_R\left[1 - \left(1 - \frac{\tau_\varepsilon}{\tau_\sigma}\right)e^{-t/\tau_\sigma}\right]H(t), \tag{5}$$

where M_R is the relaxed modulus, $H(t)$ is the Heaviside function, τ_ε and τ_σ are strain and stress relaxation times, respectively. They then generalized (4) and (5), without any derivation, to the case with *L* standard linear solids,

$$Q(\omega)=\frac{1+\sum_{l=1}^{L}\frac{\omega^2\tau_{\sigma,l}(\tau_{\varepsilon,l}-\tau_{\sigma,l})}{1+\omega^2\tau_{\sigma,l}^2}}{\sum_{l=1}^{L}\frac{\omega\tau(\tau_{\varepsilon,l}-\tau_{\sigma,l})}{1+\omega^2\tau_{\sigma,l}^2}}, \quad (6)$$

$$\psi(t)=M_R\left[1-\sum_{l=1}^{L}\left(1-\frac{\tau_{\varepsilon,l}}{\tau_{\sigma,l}}\right)e^{-t/\tau_{\sigma,l}}\right]H(t), \quad (7)$$

where $\tau_{\varepsilon,l}$ and $\tau_{\sigma,l}$ are strain and stress relaxation times, respectively, for the l-th standard solid. We should mention that they never explicitly wrote an express for $M(\omega)$ for their generalization.

Later on, using this same generalized solid model, Carcione found the following results (Carcione 07)

$$M(\omega)=\frac{M_R}{L}\sum_{l=1}^{L}\frac{1+i\omega\tau'_{\varepsilon,l}}{1+i\omega\tau'_{\sigma,l}}, \quad (8)$$

$$Q_v(\omega)=\frac{1+\frac{1}{L}\sum_{l=1}^{L}\frac{\omega^2\tau'_{\varepsilon,l}(\tau'_{\varepsilon,l}-\tau'_{\sigma,l})}{1+(\omega\tau'_{\sigma,l})^2}}{\frac{1}{L}\sum_{l=1}^{L}\frac{\omega(\tau'_{\varepsilon,l}-\tau'_{\sigma,l})}{1+(\omega\tau'_{\sigma,l})^2}}, \quad (9)$$

$$\psi(t)=M_R\left[1-\frac{1}{L}\sum_{l=1}^{L}\left(1-\frac{\tau'_{\varepsilon,l}}{\tau'_{\sigma,l}}\right)e^{-t/\tau'_{\sigma,l}}\right]H(t). \quad (10)$$

Here we have used $\tau'_{\varepsilon,l}$ and $\tau'_{\sigma,l}$ to denote the strain and stress relaxation times in these equations, and later on we should see the reason. The detail derivation of (8)-(10) can be found in Mozco and Kristek;s paper (Mozco 05). Formally, the only difference between the above two set of equations (6)-(7) and (8)-(10) is the factor *1/L*. This is the famous *1/L* factor issue mentioned in the literature.

DERIVATION OF EQUATION (6) AND (7)

In order to see how this factor *1/L* arises, we derive (6) and (7) and the modulus function $M(\omega)$ using a rheological model of generalized Maxwell Body, which is composed of L Maxwell bodies (spring δM_l and dashpot η_l in series) connected in parallel with a spring M_R, as shown schematically in Fig. 1b. This model is fully equivalent to the generalized Zener body (Moczo 05,Cao 14), as one can see by replacing the L parallel pure springs in the generalized Zener body by a single pure spring with spring constant M_R, but is more suitable to demonstrate how this *1/L* issue arises.

For the pure spring, the stress-strain relation is

$$\sigma_0(t)=M_R\varepsilon_0(t), \quad (11)$$

and for each Maxwell body l, a given stress σ_l produces a deformation $\varepsilon_{l,1}$ on the spring, and a deformation $\varepsilon_{l,2}$ on the dashpot. The stress-strain relation in the spring is given (Carcione 07)

$$\sigma_l(t)=\delta M_l\varepsilon_{l,1}(t), \quad (12)$$

and the stress-strain relation on the dashpot is

$$\sigma_l(t)=\eta_l\partial_t\varepsilon_{l,2}(t). \quad (13)$$

The total deformation on each of the Maxwell body is

$$\varepsilon_l(t)=\varepsilon_{l,1}(t)+\varepsilon_{l,1}(t). \quad (14)$$

Using (12)-(14), we obtain the stress-strain relation of each Maxwell element as

$$\frac{\partial_t\sigma_l}{\delta M_l}+\frac{\sigma_l}{\eta_l}=\partial_t\varepsilon_l. \quad (15)$$

Transforming into frequency domain, we have

$$\sigma_l(\omega)=\frac{i\omega\eta_l}{1+i\omega\tau_{\sigma,l}}, \tau_{\sigma,l}=\frac{\eta_l}{\delta M_l}. \quad (16)$$

Now the total strain on the whole generalized Maxwell body is equal to the strain on each Maxwell element,

$$\varepsilon=\varepsilon_0=\varepsilon_1=\cdots=\varepsilon_L, \quad (17)$$

and the total stress is the sum over each Maxwell element,

$$\begin{aligned}\sigma(\omega)&=\sum_{l=1}^{L}\sigma_l(\omega)\\&=M_R\left[1+\frac{1}{M_R}\sum_{l=1}^{L}\frac{i\omega\eta_l}{1+i\omega\tau_{\sigma,l}}\right]\varepsilon(\omega).\end{aligned} \quad (18)$$

This equation, as well as the definition of the stress relaxation time $\tau_{\sigma,l}$, is the same as (7) in Moczo and Kristek's paper (Mozco 05). Now we define the stress relaxation time for each Maxwell element as

$$\tau_{\varepsilon,l}=\eta_l\left(\frac{1}{M_R}+\frac{1}{\delta M_l}\right). \quad (19)$$

As we will show later, this definition of $\tau_{\varepsilon,l}$ is different from that by Carcio (Carcio 07) and Moczo and Kristek's paper (Mozco 05).

Now using (18) and (19), we obtain the complex modulus as

$$M(\omega)=M_R\left(1-L+\sum_{l=1}^{L}\frac{1+i\omega\tau_{\varepsilon,l}}{1+i\omega\tau_{\sigma,l}}\right). \quad (20)$$

We should mention that (20) for $M(\omega)$ is never explicitly shown in Liu *et al*'s paper (Liu 76). Using (20) and (2), we obtain the frequency dependent Q factor given by (6). The time-dependent relaxation function (7) is obtained by doing inverse Fourier transformation of $M(\omega)/i\omega$ using (20) for $M(\omega)$. This clearly shows the justification of generalizing (3)-(5) of the model with a single standard linear solid to (20), (6) and (7) of the model with L standard linear solids without the $1/L$ factor in front of the summation operation.

But the derivation of (8)-(10) by Moczo and Kristek (Mozco 05) is also mathematically correct. As a matter of factor, these two set of equations are actually equivalent because the stress relaxation time in (8)-(10) is simply related to that in (20), (6) and (7) by the following relation

$$\tau'_{\varepsilon,l} - \tau_{\sigma,l} = L(\tau_{\varepsilon,l} - \tau_{\sigma,l}). \qquad (21)$$

As one can verify, substituting (21) into (8)-(10), one immediately obtains (20), (6) and (7). Thus, if one uses one of the two sets of equation consistently, there should be no any error at all.

CONCLUSIONS

We have shown that the so-called missing 1/L factor in Liu, Anderson and Kanamori's paper (Liu 76) is purely caused by a different definition of the stress relaxation time, and if one uses these equations consistently, there is no error arising. We believe this clarification is important to the geophysical community.

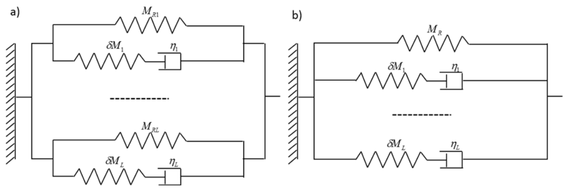

Figure 1 a) Schematic diagram of generalized Zener body composed of L parallel so-called standard linear solids in parallel. M_{Rl} and δM_{Rl} denote elastic moduli, and η_l viscosity. b) Schematic diagram of generalized Maxwell body composed of L parallel so-called Maxwell solids in parallel with a spring with a spring constant M_R. The two models are equivalent, as one can see by replacing the L parallel pure springs in the generalized Zener body by a single pure spring with an effective spring constant M_R.

REFERENCES

Liu, H.-P, D. L. Anderson, and H. Kanamori, 1976, Velocity dispersion due to anelasticity: implication for seismology and mantle composition, Geophys. J. R. Astr. Soc., **47**, 41-58.

Carcione, J. M., 2007, Wavefields in real media: Wave propagation in anisotropic, anelastic, porous and electromagnetic media: Elsevier.

Moczo, P. and J. Kristek, 2005, On the rheological models used for time domain method of seismic wave propagation. Geophys. Res. Lett., **32**, L01306.

Cao, Danping, and Xingyao Yin, 2014, Equivalent relations of generalized rheological models for viscoelastic seismic-wave modeling. Bull. Seism. Soc. Am., **104**, 260.

McDonal, F. J., and F. A. Angona *et. al.*, 1958, Attenuation of shear and compressional waves in Pierre shale: Geophysics, **23**, 421–439.

Dahlen, F., and J. Tromp, 1998, Theoretical global seismology: Princeton University Press.

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