OPTIMIZATION OF A PML MODEL FOR ELASTODYNAMICS

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Abstract

The main advantage of perfectly matched layers in the context of elastodynamics is that no reflection is generated at the interface between the medium and the absorbing layers. Unfortunately this property is lost after discretization. In this work, we compute the analytical expression of the discrete reflection coefficients in the context of a 2D finite difference scheme, and then minimize these coefficients based upon a least-square approach.

Introduction

Collino and Tsogka [1] developed a perfectly matched layer (PML) model based on the elastodynamics equation written as a first-order system in velocity and stress. One of the most attractive properties of a PML model is that no reflection occurs at the interface between the medium and the absorbing layer. Therefore, the layer does not send spurious energy back into the medium. However, the layer must be truncated in order to be able to perform numerical simulations, and such truncation may create a reflected wave whose amplitude can be amplified by the discretization process. In this work, we analyze the effect of the truncated layer on the solution by computing the analytical expression of the discrete reflection coefficients after discretization by the finite difference scheme of Virieux [3]. The calculations are uneasy because we have to account for the conversions of waves (from pressure - P to shear - S, or shear to pressure). The results demonstrate that indeed the truncated layer generates a reflected wave. The amplitude of this spurious wave can then be minimized based upon a least-square approach similar to that developed in Collino and Monk [2] in the context of Maxwell's equations.

1 A PML model for elastodynamics

We study the equations of elastodynamics formulated in velocity and stress. This formulation has been used by F. Collino and C. Tsogka [1], and in this study we use the same PML model as that used by these authors. We restrict ourselves to the two-dimensional case. Let v and σ respectively denote the velocity vector field and the stress tensor, the elastodynamics system car be written as:

$$\begin{cases} \rho\partial_t v - \mathbf{div}\sigma = 0, \\ H^{-1}\partial_t\sigma - \epsilon(v) = 0, \end{cases} \tag{1}$$

where ρ denotes density, and the 2×2 matrix $\epsilon(v)$ is the strain tensor defined as:

$$\epsilon_{ij}(v) = \frac{1}{2}(\partial_j v_i + \partial_i v_j).$$

The stress tensor is related to the strain tensor by Hooke's law:

$$\sigma = \sigma(v)(x,t) = H(x)\epsilon(v)(x,t).$$

The PML model that we use is based on a decomposition of the unknown v and σ fields into split components

$$v = v^{\perp} + v^{\|}, \qquad \sigma = \sigma^{\perp} + \sigma^{\|},$$

and we then introduce in the system a damping coefficient that is a function of the coordinate normal to the absorbing boundary. The modified equations are those that contain normal derivatives with respect to the edge of the absorbing layer. In what follows, we consider a flat layer located in $x = 0$. We also assume that the medium is homogeneous, elastic and isotropic. The PML model can then be written as:

$$\rho(\partial_t + d(x))v_x^{\perp} = \partial_x\sigma_{xx} \quad ; \; \rho\partial_t v_x^{\|} = \partial_y\sigma_{xy},$$

$$\rho(\partial_t + d(x))v_y^{\perp} = \partial_x\sigma_{xy} \quad ; \; \rho\partial_t v_y^{\|} = \partial_y\sigma_{yy},$$

$$(\partial_t + d(x))\sigma_{xx}^{\perp} = (\lambda + 2\mu)\partial_x v_x \quad ; \; \partial_t\sigma_{xx}^{\|} = \lambda\partial_y v_y,$$

$$(\partial_t + d(x))\sigma_{yy}^{\perp} = \lambda\partial_x v_x \quad ; \; \partial_t\sigma_{yy}^{\|} = (\lambda + 2\mu)\partial_y v_y,$$

$$(\partial_t + d(x))\sigma_{xy}^{\perp} = \mu\partial_x v_y \quad ; \; \partial_t\sigma_{xy}^{\|} = \mu\partial_y v_x,$$

where we have written the symmetric tensor σ as the vector ${}^t(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$.

2 The discrete PML model

We now use the finite-difference scheme introduced by Virieux [3] in order to discretize the above model. Suppressing the time variable, the discrete model can be written as:

$$\begin{cases} (v_x)_{l,j} = (v_x^{\perp})_{l,j} + (v_x^{\|})_{l,j}, \\ (i\omega + d_l)(v_x^{\perp})_{l,j} = \dfrac{(\sigma_{xx})_{l+\frac{1}{2},j} - (\sigma_{xx})_{l-\frac{1}{2},j}}{\rho h}, \\ i\omega(v_x^{\|})_{l,j} = \dfrac{(\sigma_{xy})_{l,j+\frac{1}{2}} - (\sigma_{xy})_{l,j-\frac{1}{2}}}{\rho h}; \end{cases}$$

$$\begin{cases} (v_y)_{l+\frac{1}{2},j+\frac{1}{2}} = (v_y^{\perp})_{l+\frac{1}{2},j+\frac{1}{2}} + (v_y^{\|})_{l+\frac{1}{2},j+\frac{1}{2}}, \\ (i\omega + d_{l+\frac{1}{2}})(v_y^{\perp})_{l+\frac{1}{2},j+\frac{1}{2}} = \dfrac{(\sigma_{xy})_{l+1,j+\frac{1}{2}} - (\sigma_{xy})_{l,j+\frac{1}{2}}}{\rho h}, \\ i\omega(v_y^{\|})_{l+\frac{1}{2},j+\frac{1}{2}} = \dfrac{(\sigma_{yy})_{l+\frac{1}{2},j+1} - (\sigma_{yy})_{l+\frac{1}{2},j}}{\rho h}; \end{cases}$$

$$\begin{cases} (\sigma_{xx})_{l+\frac{1}{2},j} = (\sigma_{xx}^{\perp})_{l+\frac{1}{2},j} + (\sigma_{xx}^{\|})_{l+\frac{1}{2},j}, \\ (i\omega + d_{l+\frac{1}{2}})(\sigma_{xx}^{\perp})_{l+\frac{1}{2},j} = (\lambda + 2\mu)\dfrac{(v_x)_{l+1,j} - (v_x)_{l,j}}{h}, \\ i\omega(\sigma_{xx}^{\|})_{l+\frac{1}{2},j} = \lambda\dfrac{(v_y)_{l+\frac{1}{2},j+\frac{1}{2}} - (v_y)_{l+\frac{1}{2},j-\frac{1}{2}}}{h}; \end{cases}$$

$$\begin{cases} (\sigma_{yy})_{l+\frac{1}{2},j} = (\sigma_{yy}^{\perp})_{l+\frac{1}{2},j} + (\sigma_{yy}^{\|})_{l+\frac{1}{2},j}, \\ (i\omega + d_{l+\frac{1}{2}})(\sigma_{yy}^{\perp})_{l+\frac{1}{2},j} = \lambda\dfrac{(v_x)_{l+1,j} - (v_x)_{l,j}}{h}, \\ i\omega(\sigma_{yy}^{\|})_{l+\frac{1}{2},j} = (\lambda + 2\mu)\dfrac{(v_y)_{l+\frac{1}{2},j+\frac{1}{2}} - (v_y)_{l+\frac{1}{2},j-\frac{1}{2}}}{h}; \end{cases}$$

$$\begin{cases} (\sigma_{xy})_{l,j+\frac{1}{2}} = (\sigma_{xy}^{\perp})_{l,j+\frac{1}{2}} + (\sigma_{xy}^{\|})_{l,j+\frac{1}{2}}, \\ (i\omega + d_l)(\sigma_{xy}^{\perp})_{l,j+\frac{1}{2}} = \mu\dfrac{(v_y)_{l+\frac{1}{2},j+\frac{1}{2}} - (v_y)_{l-\frac{1}{2},j+\frac{1}{2}}}{h}, \\ i\omega(\sigma_{xy}^{\|})_{l,j+\frac{1}{2}} = \mu\dfrac{(v_x)_{l,j+1} - (v_x)_{l,j}}{h}. \end{cases}$$

The parameter h represents the size of a mesh grid cell in the two spatial directions ($h = \Delta x = \Delta y$).

3 Calculation of the reflection coefficients

In order to analyze the influence of the absorbing layer and to ultimately optimize its behavior, we now compute the analytical expression of the discrete PML reflection coefficients. We make use of a plane wave analysis, which implies that we look for particular solutions of the discrete problem of the form:

$$(V^k)_{l+\frac{1}{2},j+\frac{1}{2}} = (\hat{V}^k)_{l+\frac{1}{2}} \exp[-ik_y(j + \frac{1}{2})h + i\omega t], \quad (2)$$

with

$$V^k = \begin{bmatrix} v_x^k \\ v_y^k \end{bmatrix}, \qquad k = \perp, \|$$

and

$$(\hat{V})_{l+\frac{1}{2}} = (\hat{V}^{\perp})_{l+\frac{1}{2}} + (\hat{V}^{\|})_{l+\frac{1}{2}}.$$

After getting rid of σ in the discrete scheme, we use expressions (2) in the equations obtained, and get:

$$\begin{aligned} (\hat{v}_x)_l = {} & \frac{\lambda + 2\mu}{\rho h^2(i\omega + d_l)} \left(\frac{(\hat{v}_x)_{l+1} - (\hat{v}_x)_l}{i\omega + d_{l+\frac{1}{2}}} - \frac{(\hat{v}_x)_l - (\hat{v}_x)_{l-1}}{i\omega + d_{l-\frac{1}{2}}} \right) \\ & + 2\sin(\frac{k_y h}{2}) \frac{\lambda + \mu}{\omega\rho h^2(i\omega + d_l)} \left((\hat{v}_y)_{l-\frac{1}{2}} - (\hat{v}_y)_{l+\frac{1}{2}} \right) \\ & + 4\sin^2(\frac{k_y h}{2}) \frac{\mu}{\rho\omega^2 h^2} (\hat{v}_x)_l, \end{aligned} \quad (3)$$

$$\begin{aligned} (\hat{v}_y)_{l+\frac{1}{2}} = {} & \frac{\mu}{\rho h^2(i\omega + d_{l+\frac{1}{2}})} \left(\frac{(\hat{v}_y)_{l+\frac{3}{2}} - (\hat{v}_y)_{l+\frac{1}{2}}}{i\omega + d_{l+1}} - \frac{(\hat{v}_y)_{l+\frac{1}{2}} - (\hat{v}_y)_{l-\frac{1}{2}}}{i\omega + d_l} \right) \\ & + 2\sin(\frac{k_y h}{2}) \frac{\lambda + \mu}{\omega\rho h^2(i\omega + d_{l+\frac{1}{2}})} \left((\hat{v}_x)_l - (\hat{v}_x)_{l+1} \right) \\ & + 4\sin^2(\frac{k_y h}{2}) \frac{\lambda + 2\mu}{\rho\omega^2 h^2} (\hat{v}_y)_{l+\frac{1}{2}}. \end{aligned} \quad (4)$$

We then look for particular solutions of (3) and (4) of the form:

- P waves:

$$(\hat{V})_q = \vec{d}_p e^{-ik_x qh} + R_{pp}\vec{d}_p^r e^{ik_x qh} + R_{ps}\vec{d}_s^r e^{ik_x qh} \qquad \text{for } q = l, l + \frac{1}{2} \leq 0,$$

$$(\hat{V})_q = T_{pp}\vec{d}_p^t e^{-i\tilde{k}_x qh} + T_{ps}\vec{d}_s^t e^{-i\tilde{k}_x qh} \qquad \text{for } q = l, l + \frac{1}{2} \geq 0,$$

with

$$\vec{d}_p = {}^t(\cos\theta, \sin\theta),\ \vec{d}_p^r = {}^t(-\cos\theta, \sin\theta),$$
$$\vec{d}_s^r = {}^t(-\sin\theta_2, \cos\theta_2),\ \vec{d}_p^t = {}^t(\cos\theta_3, \sin\theta_3),$$
$$\vec{d}_s^t = {}^t(-\sin\theta_4, \cos\theta_4);$$

- S waves:

$$(\hat{V})_q = \vec{d}_s e^{-ik_x qh} + R_{ss}\vec{d}_s^r e^{ik_x qh} + R_{sp}\vec{d}_p^r e^{ik_x qh} \qquad \text{for } q = l, l + \frac{1}{2} \leq 0,$$

$$(\hat{V})_q = T_{ss}\vec{d}_s^t e^{-i\tilde{k}_x qh} + T_{sp}\vec{d}_p^t e^{-i\tilde{k}_x qh} \qquad \text{for } q = l, l + \frac{1}{2} \geq 0,$$

with

$$\vec{d_s} = {}^t(-\sin\theta, \cos\theta),\ \vec{d_s^r} = {}^t(\sin\theta, \cos\theta),$$
$$\vec{d_p^r} = {}^t(\cos\theta_2, \sin\theta_2),\ \vec{d_s^t} = {}^t(-\sin\theta_3, \cos\theta_3),$$
$$\vec{d_p^t} = (\cos\theta_4, -\sin\theta_4).$$

We then determine the reflection coefficients for an infinite absorbing layer as well as for a layer of finite size. Here, we only present the finite size case, which can be described by:
$(d_{\frac{1}{2}}, \dots, d_{n_l-\frac{1}{2}})$ et $(d_1, \dots, d_{n_l})$,
where n_l represents the number of grid points in the layer in the x direction; $h \times n_l$ is then the thickness of the layer. We impose homogeneous Dirichlet condition at the end of the absorbing layer: $(\hat{v}_x)_{n_l} = (\hat{v}_y)_{n_l+\frac{1}{2}} = 0$. Let us denote:

$$a_l = (i\omega + d_{l-1})(h^2 - 4\frac{\mu}{\rho\omega^2}\sin^2(\frac{k_y h}{2})) - (b_l + b_{l-1}),$$
$$b_l = -\frac{\lambda + 2\mu}{\rho(i\omega + d_{l-\frac{1}{2}})},$$
$$c_l = (i\omega + d_{l-\frac{1}{2}})(h^2 - 4\frac{\lambda + 2\mu}{\rho\omega^2}\sin^2(\frac{k_y h}{2})) - (e_l + e_{l-1}),$$
$$e_l = -\frac{\mu}{\rho(i\omega + d_l)},$$
$$\alpha = 2\sin(\frac{k_y h}{2})\frac{\lambda + \mu}{\omega\rho},\ \gamma = \frac{\lambda + 2\mu}{i\omega\rho}(\hat{v}_x)_{-1} + \alpha(\hat{v}_y)_{-\frac{1}{2}},$$
$$\delta = \frac{\mu}{i\omega\rho}(\hat{v}_y)_{-\frac{1}{2}};$$

we can then show that the discrete reflection coefficients R_{pp}, R_{ps}, R_{ss} et R_{sp} are given by:

$$\begin{aligned} R_{pp} = \cos(\theta + \theta_2)^{-1}\Big[&\cos(\theta - \theta_2) \\ &+ \cos\theta_2\, {}^tF\mathcal{K}^{-1}\mathcal{M}^{-1}(\delta\mathcal{A}\mathcal{N}^{-1} - \gamma Id)F \\ &- \sin\theta_2 \exp(i\tilde{k}_x\frac{h}{2})\, {}^tF\mathcal{L}^{-1}\mathcal{N}^{-1}(\gamma\, {}^t\mathcal{A}\mathcal{M}^{-1} + \delta Id)F\Big], \end{aligned}$$

$$\begin{aligned} R_{ps} = \cos(\theta + \theta_2)^{-1}\Big[&-\sin 2\theta \\ &+ \cos\theta \exp(i\tilde{k}_x\frac{h}{2})\, {}^t\mathcal{F}\mathcal{L}^{-1}\mathcal{N}^{-1}(\delta Id + \gamma\, {}^t\mathcal{A}\mathcal{M}^{-1})F \\ &+ \sin\theta\, {}^tF\mathcal{K}^{-1}\mathcal{M}^{-1}(\gamma Id - \delta\mathcal{A}\mathcal{N}^{-1})F\Big], \end{aligned}$$

$$\begin{aligned} R_{ss} = \cos(\theta + \theta_2)^{-1}\Big[&-\cos(\theta - \theta_2) \\ &+ \cos\theta_2 \exp(i\tilde{k}_x\frac{h}{2})\, {}^tF\mathcal{L}^{-1}\mathcal{N}^{-1}(\delta Id + \gamma\, {}^t\mathcal{A}\mathcal{M}^{-1})F \\ &+ \sin\theta_2\, {}^tF\mathcal{K}^{-1}\mathcal{M}^{-1}(\delta\mathcal{A}\mathcal{N}^{-1} - \gamma Id)F\Big], \end{aligned}$$

$$\begin{aligned} R_{sp} = \cos(\theta + \theta_2)^{-1}\Big[&\sin 2\theta \\ &+ \cos\theta\, {}^tF\mathcal{K}^{-1}\mathcal{M}^{-1}(\gamma Id - \delta\mathcal{A}\mathcal{N}^{-1})F \\ &- \sin\theta \exp(i\tilde{k}_x\frac{h}{2})\, {}^tF\mathcal{L}^{-1}\mathcal{N}^{-1}(\gamma\, {}^t\mathcal{A}\mathcal{M}^{-1} + \delta Id)F\Big]; \end{aligned}$$

with:

$$\mathcal{M} = \begin{bmatrix} a_1 & b_1 & 0 & & 0 \\ b_1 & \ddots & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & & \ddots & \ddots & b_{n_l-1} \\ 0 & & 0 & b_{n_l-1} & a_{n_l} \end{bmatrix}$$

$$\mathcal{N} = \begin{bmatrix} c_1 & e_1 & 0 & & 0 \\ e_1 & \ddots & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & & \ddots & \ddots & e_{n_l-1} \\ 0 & & 0 & e_{n_l-1} & c_{n_l} \end{bmatrix},$$

$$\mathcal{A} = \begin{bmatrix} \alpha & 0 & & 0 \\ -\alpha & \ddots & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & -\alpha & \alpha \end{bmatrix}, F = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\mathcal{K} := Id + \mathcal{M}^{-1}\mathcal{A}\mathcal{N}^{-1}\, {}^t\mathcal{A} \quad \text{and}$$
$$\mathcal{L} := Id + \mathcal{N}^{-1}\, {}^t\mathcal{A}\mathcal{M}^{-1}\mathcal{A}$$

4 Optimization of the reflection coefficients

We can then use the analytical expression of the reflection coefficients in order to optimize the damping profile in the PML layer based upon a least-square approach used by other authors for Maxwell's equations [2]. One of the main difficulties is that we have to simultaneously optimize four different reflection coefficients. Numerical tests are currently being performed.

References

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